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curriculum S.12A, adaptation

A SUGGESTED ADAPTATION OF CURRICULUM S.12A

mathematics grade 11

ONTARIO DEPARTMENT OF EDUCATION

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PREFACE

This interim statement is intended for use in the Five-year Program in conjunction with the present Grade 11 course of study in Mathematics (Curriculum S.12A). It should be of particular value to those teachers planning a program to follow the new course of study in Grade 10 (Curriculum I.12D) until the revised Mathematics course in the Five-year Program for Grade 11 is available. It will be followed by a similar statement for Grade 12.

When S.12A was first introduced in 1964 most students entering Grade 11 had little or no experience with ideas such as number structure, the language of sets, relations and functions, and graphing in the Cartesian Plane; and so at that time it was necessary to prescribe an introductory treatment of these topics in S.12A. Since then, however, students in the Intermediate Division, through the revised courses in Grade 7 (I.12A), Grade 8 (I.12B), Grade 9 (I.12C), and Grade 10 (I.12D), have become familiar with these concepts, as well as with new concepts such as vectors, transformations, and groups. These changes in the program to the end of Grade 10 make it advisable that modifications be made in the content of and the approach implied by Curriculum S.12A for 1968-69.

Since the Curriculum Committee which is revising the Mathematics courses in the Five-year Program for Grades 11 and 12 will not complete its recommendations in time for September of 1968, this statement has been designed to <u>supplement</u> Curriculum S.12A in the interim period.

SUGGESTIONS FOR AN ADAPTATION OF S.12A, SEQUENTIAL TO I.12D

General Comments

1. This adaptation suggests those topics now in S.12A which might be omitted, modified, or extended on the basis of the students' previous experiences, and supplements this with suggestions for (i) extending and applying "new" concepts (such as vectors and transformations) from earlier grades, and (ii) introducing new concepts.

   Teachers should make a selection of the suggestions which best suit the needs of their students. They should be governed in this selection by the knowledge that there is not sufficient time to explore all the suggested adaptations, and by the availability of reference material to supplement their lesson planning. It is further suggested that topics from S.12B not be considered in Grade 11 at this time.

2. Only Unit 1 of S.12A has been rewritten; modifications have been suggested for the other units.

3. Formal units on logic and sets have not been included in this statement. However, the language, notation, and ideas associated with these concepts should be integrated with the discussion of the other topics in the course, whenever this will help to clarify an understanding of these topics. *

   The teacher should ensure that the students are familiar with the use of the connectives:

   and, or, not, if ... then ..., only if, if and only if (iff);

   the meaning of:

   conditional, implication, converse, bi-conditional, contrapositive, inverse, set, subset, union, intersection, universal and existential quantification (at an intuitive level);

A 1
B 1
C ch.A
D (p.b. Nature o Proof)
E 12
F 2
G 11
N 1
O 5
$X_{alg.1}$
Y 2
EE 8
GG
TT
UU 1

* See 5. on page 3.

and the use of symbols such as:

$\{\ \}$, $\in$, $\notin$, $\subset$, $\subseteq$, $\cup$, $\cap$, and $\emptyset$.

The student has likely had some prior experience at an intuitive level with most of these ideas. The objective now should be to consolidate these experiences as the occasions permit.

4. Consideration should be given to starting the year with Geometry (Units 2., 3., and 4.) supplemented by the methods of vectors and transformations for alternate proofs of deductions. This will establish continuity with the units on Geometry, Vectors, and Transformations which have likely been discussed in the spring of Grade 10, and will avoid starting the year once more with a review of number structure. By the time the Geometry units are completed the student will be ready for a review and extension of his knowledge of the real numbers. The unit on Operations with Real Numbers leads more naturally into the study of Linear Functions, Analytic Geometry of the Straight Line, and the Trigonometric Functions.

5. This statement has been prepared with a view to utilizing the present Grade 11 textbooks, wherever possible. However, many of the suggestions will require supplementary reference books. A list of references is given at the end of this statement on pages 44 to 46; references for specific suggestions are given in the right margin by a coding such as D7, which means reference text D chapter 7.

6. Discussion of some of the optional topics from the Grade 10 (I.12D) Course of Study might also be considered.

Unit 1 OPERATIONS IN THE SET OF REAL NUMBERS

(a suggested replacement for Unit 1 of S.12A)

Comment

Many of the topics in this unit are also included in the I.12 sequence of courses of study, in most cases at an intuitive level. They have been included here, since some review and consolidation may be necessary before these topics can be extended. Teachers should determine the extent of the review needed, and adapt the treatment of this unit according to the needs of their students.

Course Outline		Suggestions
1.* Rational Numbers:		
(i) a review of the definitions of addition and multiplication, group and field properties	A B C ch.B D 1 E 1 F 1 G 1 AA 3 II 1	By recalling the fractional form $\frac{a}{b}$ of a rational number and the definitions of addition and multiplication, and by assuming knowledge of the properties of integers, it can be observed that the Rationals form a group under +, and the non-zero Rationals a group under ×, that the commutative property holds for + and ×, and the distributive property for × over +. The field properties of the Rationals are thus established.
	O 3 U 9 AA 1,10 EE 1	This review might also include other examples of groups, binary operations, and finite systems.
(ii) order relation, density		These topics are discussed in the present Grade 10 and Grade 11 texts, although they are not specifically referred to in I.12D.

* In this statement, Q will be used to represent the set of rational numbers.

Course Outline		Suggestions
(iii) addition, subtraction, multiplication, factorization, and division with rational expressions	H 11 I 4,5 J 6,7 V 7 $X_{alg.}$ 3 AA 3 DD 17 HH 3,4 II 3	Operations with rational expressions, and factorization are included in most mathematics texts used in recent years in Ontario. This topic is intended to give the student a broader understanding of the operations as applied to rational expressions. If extensive practice is needed to improve the students' facility with these manipulations, it should be given in appropriate short intervals throughout the year, rather than all at one time. This continued use of algebraic skills avoids the risk of rote learning and subsequent boredom. This topic should be extended into Grade 12 as needed, at which time the factor theorem, incomplete square, sum and difference of cubes might be more meaningfully discussed.
2. Irrational Numbers:		
(i) review of the periodic and non-periodic decimal forms of rational and irrational numbers, respectively		The completeness property is significant in the discussion of graphs of functions with domain the "Reals", as it is fundamental to an understanding of continuity, and later of discontinuity. In discussing the order relation on the real numbers, the use of decimal approximations for rational and irrational numbers permits a determination of "is greater than". The need for "ordering" of real numbers could be illustrated by problems such as:
(ii) discussion that an irrational number can be assigned a position on a number line		
(iii)* the set of real numbers defined as the union of the set of rational numbers and the set of irrational numbers, assumption of the completeness property, order relation on the real numbers	A 2 B 2 C 1 II 5	• determining the relative position of 2 numbers on a number line • determining if an angle of a triangle is right, acute, or obtuse by comparing $a^2 + b^2$ with c^2, where $a,b,c \in R$ and represent the measures of the three sides • determining if a point (a,b) is above, on, or below a line defined by $y = mx + k$ • problems from science.

* In this statement, R will be used to represent the set of real numbers.

Course Outline

3. Radicals:

(i) a review of operations with radicals (as outlined in I.12D, Unit 1, Section 3), including rationalization of denominators of expressions such as

$$\frac{3\sqrt{2} + 5\sqrt{3}}{\sqrt{3} + \sqrt{2}}$$

A 2
B 2
C 3
II 6

Suggestions

This topic allows for a review and a possible extension in depth of the manipulative skills with radical expressions.

(ii) proof that numbers such as $\sqrt{2}$, $\sqrt{3}$ and $(\sqrt{2} + \sqrt{3})$ are not rational, discussion that the set of irrational numbers is not closed and hence does not form a field.

A 2
B 2
C 1
DD 8
II 5

Proofs to show that $\sqrt{2}$, for example, is not rational are included in the present Grade 11 texts. This topic offers a good opportunity to illustrate the indirect method of proof. The proof that $(\sqrt{2} + \sqrt{3})$ is irrational, also can employ the indirect method, as follows.

Let $(\sqrt{2} + \sqrt{3}) = x$, where $x \in R$.
Either $x \in Q$ or $x \notin Q$.
Suppose $x \in Q$, then $5 + 2\sqrt{6} = x^2$, and $x^2 \in Q$.
Hence $2\sqrt{6} = x^2 - 5$, and $(x^2 - 5) \in Q$.
So $2\sqrt{6} \in Q$, thus $\sqrt{6} \in Q$, which is false.
The assumption $x \in Q$ must be false, and it follows that $x \notin Q$.
Thus $(\sqrt{2} + \sqrt{3})$ is irrational.

(iii) review of the field with elements of the form

$a + b\sqrt{2}$, where $a, b \in Q$

D 3
E 2
F 3
G 2
PP

Although this topic appears in I.12D, its inclusion in Unit 1 offers the opportunity of reviewing the properties of a field, while practicing some of the techniques of (i).

Course Outline		Suggestions
(iv) definition of absolute value, simplification of radicals such as: $\sqrt{x^2}$, $\sqrt{(-5)^2}$, $\sqrt{x^2 - 2x + 1}$, $\sqrt{x^2 + 2x + 1}$.	A 2	Absolute value is included in I.12C, but has not been mentioned in I.12D (although included in the texts). The definition: for real x,
	C ch B,2	$\lvert x\rvert = \begin{cases} x & \text{iff } x \geq 0 \\ -x & \text{iff } x < 0 \end{cases}$
		should be discussed, and reinforced by examining simplification of radicals (as suggested) and the solution sets (graphs) of sentences such as:
	RR	$\lvert x\rvert = 3$, $\lvert x + 3\rvert = 2$, $\lvert x\rvert > 3$, $\lvert x\rvert < 3$, $\lvert x - 2\rvert \leq 3$, and $\lvert 2x + 3\rvert \geq 1$.
		Much of this appears in present Grade 11 texts.
(v) solution of equations such as: $\sqrt{3x - 5} = 7$, $\sqrt{3x} + 8 = 5$, $\sqrt{5x + 3} = 2\sqrt{x + 7}$, $\sqrt{x^2 + 2x + 1} = 5$, $\frac{\sqrt{x} + 16}{\sqrt{x} + 4} = \frac{\sqrt{x} + 29}{\sqrt{x} + 11}$.	A 3 B 2 C 3 II 8	This topic also is included in the present Grade 11 texts.

GENERAL COMMENTS ON UNITS 2, 3, AND 4

The comments and suggestions that follow are intended to be used in conjunction with the outlines of Units 2, 3, and 4 of S.12A. They provide ways of modifying and supplementing these units.

1. Many of the geometric properties in these units have been examined in earlier courses by inductive, and/or informal deductive methods. The same suggestions as given on pages 15 and 16 of I.12D (regarding the approach to be used for proofs by the deductive method) should be applied here. It is intended that a non-rigorous approach to proving deductions be followed, and that structure involving a sequence of undefined terms, postulates, definitions, and theorems <u>not be stressed</u>. However, the student should realize that the validity of each property proved is contingent on the validity of his hypothesis and the validity of the authorities he assumes.

D 9-11
E 7-9
F 8
G 7,8
O 10

2. The basic principles of (geometric) vectors and/or transformations in the geometric sense might be reviewed and extended as needed, in order that these concepts can be used as a basis for proving theorems and deductions. Each of these approaches can be used to supplement the more traditional "Euclidean" methods, and to provide the student with a broader experience in developing proofs using all the resources available. The student should assess the available methods, with a view to developing the simplest proof.

D 12,13
E 10,11
F 9,10
G 9,10
X_{geo}.1,2
AA 9
JJ
MM
OO

3. Consideration should also be given to investigating the extension of some of the properties of plane figures to those for the corresponding solid figures. It is important that students gain more experience in working with three-dimensional figures and problems related to them.

A 13
B 5
C 6
U 8
Y 15
LL

Unit 2 CONGRUENCE AND PARALLELISM

Suggestions for Adaptation

1. Most of the ideas in Sections 1 and 2 of S.12A have already been included in the I.12 sequence, and likely do not need any formal treatment. Section 3 should be discussed with a view to:

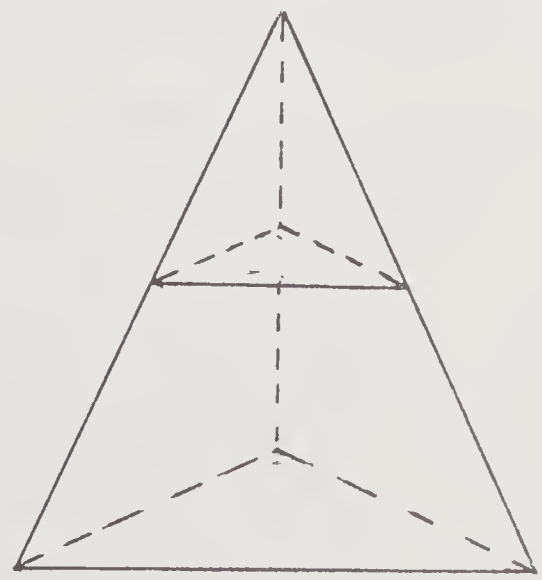
   - recalling significant geometric properties
   - investigating some new deductions and theorems
   - introducing theorems involving inequalities
   - using vector and transformation methods for proofs of some deductions
   - extending some of the deductions to their three-dimensional counterparts.

   In Section 4 the notion of a locus as a set of points should be stressed. The topics of Section 5 should be discussed, since they offer the student inductive experiences with three-dimensional figures.

A 4,5
B 3,4
C 4,5
D 9,10,11
E 7,8,9
F 8
G 7,8
S 3,4
$X_{geo.}$1

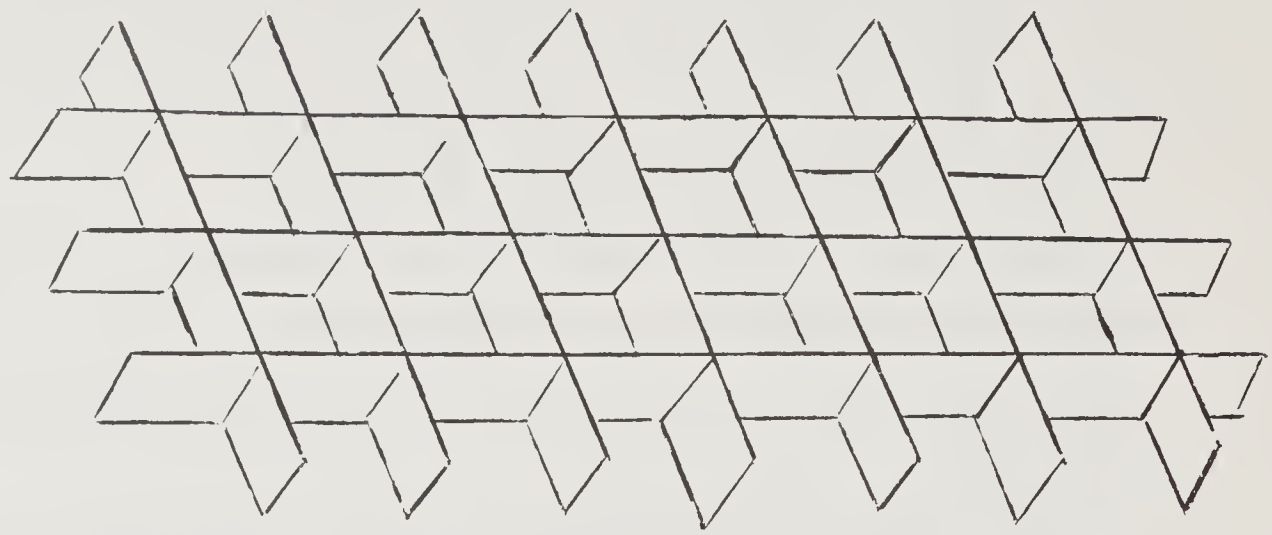
S 8
Y 15
II 4

2. As an example of a theorem with a three-dimensional counterpart, consider the theorem involving the line segment joining the mid-points of 2 sides of a triangle, then investigate the triangle (polygon) with vertices the mid points of the lateral edges of a triangular-(polygonal-) based pyramid. The area of this triangle (polygon) could be investigated in Unit 3, and the theorems generalized in Unit 4 for points dividing the lateral edges in the same ratio. Three-dimensional models could be built by students, in order to assist them in making conjectures about possible theorems.



Y 15

The transversal parallel line theorem, extended to dihedral angles in three-dimensions, can be illustrated effectively by the cardboard dividers used in packaging tinned goods.



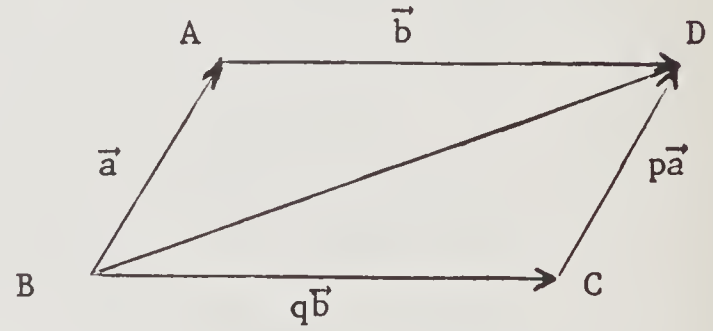
The properties dealing with angles and diagonals of a parallelopiped could be investigated.

3. This section contains some deductions and theorems, which may be solved by vector methods based on concepts introduced in I.12D. Some solutions, using vector methods, are included in order to illustrate some of the techniques, and the power of these methods. Many of these proofs utilize the fact that <u>if scalar multiples of two non-collinear, non-zero vectors are equal, then the scalar factors are each zero</u>. In Section 4, on pages 13 to 15, a summary is given of definitions and properties of vectors that are useful in proving deductions.

D 12
E 11
F 9
G 10
N 5,6
O 11
W 4,5
$X_{geo.}$ 2,3
Z 11
CC 12
DD 6
EE 3

(i) <u>In a parallelogram</u>:

(a) the opposite sides are equal



<u>Proof</u>:

ABCD is a parallelogram, thus

if $\overrightarrow{BA} = \vec{a}$, then $\overrightarrow{CD} = p\vec{a}$ for some real p, and

if $\overrightarrow{AD} = \vec{b}$, then $\overrightarrow{BC} = q\vec{b}$ for some real q.

Thus $\overrightarrow{BD} = \vec{a} + \vec{b}$, and $\overrightarrow{BD} = q\vec{b} + p\vec{a}$. (vector addition)

Hence $\vec{a} + \vec{b} = q\vec{b} + p\vec{a}$,

i.e. $(1-p)\vec{a} = (q-1)\vec{b}$.

Since $\vec{a}$, $\vec{b}$ are non-collinear,

$1 - p = 0$ and $q - 1 = 0$,

i.e. $p = q = 1$.

It follows that $\overrightarrow{BA} = \overrightarrow{CD}$ and $\overrightarrow{AD} = \overrightarrow{BC}$,

i.e. AB = CD and AD = BC.

Note: The meaning of AB should be determined from its context. It could mean (i) the line through A and B, (ii) the line segment AB (sometimes denoted $\overline{AB}$), or (iii) the measure of the line segment AB (as above, sometimes denoted $|AB|$ or $m(AB)$).

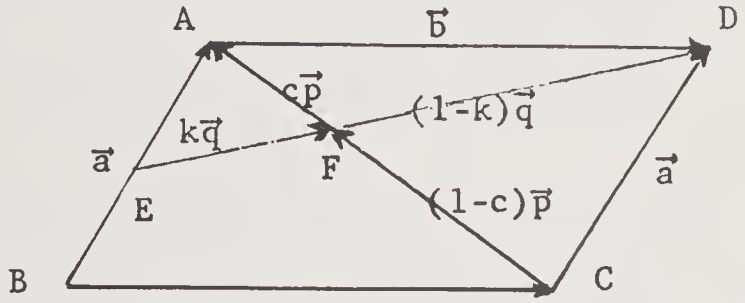
(b) the diagonals bisect each other

(c) the line segments joining a pair of opposite vertices to the mid points of a pair of opposite sides are equal and parallel

(d) a line segment joining a vertex to the mid-point of a non-adjacent side and a diagonal trisect each other.

Proof:

Vectors may be labelled as in the diagram, for some real c, k.



From ΔAEF, $\frac{1}{2}\vec{a} = k\vec{q} + c\vec{p}$. (1)

From ΔDFC, $\vec{a} = (1-c)\vec{p} + (1-k)\vec{q}$. (2)

From (1) and (2), $2k\vec{q} + 2c\vec{p} = (1-c)\vec{p} + (1-k)\vec{q}$,

i.e. $(3c-1)\vec{p} = (1-3k)\vec{q}$.

Since $\vec{p}$ and $\vec{q}$ are non-collinear, $3c-1 = 0$ and $1-3k = 0$,

i.e. $c = \frac{1}{3} = k$.

Thus $\overrightarrow{CF} = \frac{2}{3}\vec{p}$, $\overrightarrow{FA} = \frac{1}{3}\vec{p}$, $\overrightarrow{EF} = \frac{1}{3}\vec{q}$, $\overrightarrow{FD} = \frac{2}{3}\vec{q}$,

and it follows that ED and CA trisect each other.

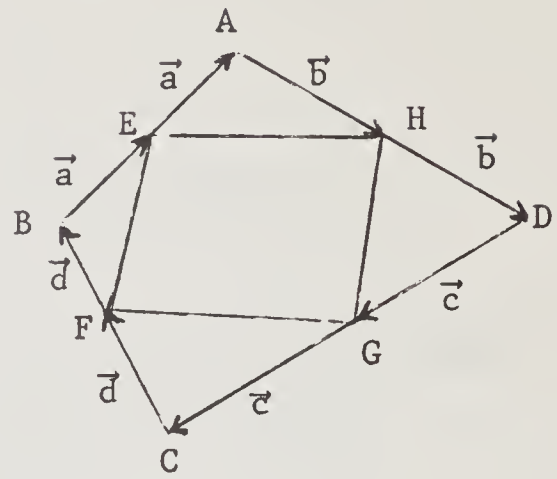
Corollary: the medians of a triangle trisect each other and are concurrent (hint: join BD).

(ii) In a quadrilateral

(a) if two opposite sides are equal and parallel, the quadrilateral is a parallelogram

(b) the line segments joining the mid points of each pair of adjacent sides form a parallelogram

Proof:



Vectors may be labelled as shown.

$\overrightarrow{EH} + \overrightarrow{HG} + \overrightarrow{GF} + \overrightarrow{FE} = \vec{0}$.

$(\vec{a}+\vec{b}) + (\vec{b}+\vec{c}) + (\vec{c}+\vec{d}) + (\vec{d}+\vec{a}) = \vec{0}$.

$2(\vec{a} + \vec{b} + \vec{c} + \vec{d}) = \vec{0}$.

It follows that

(i) $\vec{a} + \vec{b} = -(\vec{c} + \vec{d})$,

i.e. $\overrightarrow{EH} = \overrightarrow{FG}$, and so EH = FG and EH $\parallel$ FG.

(ii) $\vec{d} + \vec{a} = -(\vec{b} + \vec{c})$,

i.e. $\overrightarrow{FE} = \overrightarrow{GH}$, and so FE = GH and FE $\parallel$ GH.

Alternate Proof: Join CA.

From ΔFBE, $\overrightarrow{FE} = \vec{d} + \vec{a}$.

From ΔCBA, $\overrightarrow{CA} = (\vec{d} + \vec{d}) + (\vec{a} + \vec{a})$

$= 2(\vec{d} + \vec{a})$

$= 2\ \overrightarrow{FE}$.

Thus, CA = 2FE and CA $\parallel$ FE.

Similarly CA = 2GH and CA $\parallel$ GH.

It follows that FE = GH, FE $\parallel$ GH, and so FEHG is a parallelogram.

Corollary: In a quadrilateral based prism, the mid-points of the edges of the bases are the vertices of a parallelopiped.

(iii) If two line segments bisect each other the line segments joining their end points are equal and parallel in pairs.

(iv) A line segment joins the mid points of two sides of a triangle if and only if it is parallel to the third side and equal to one half of it.

AA 9

(v) A line through the mid point of one side of a triangle and parallel to a second side bisects the third side.

(vi) P is any point in the interior of ΔABC, and Q, R, S, and T are the mid points of AB, BP, PC, and CA respectively. Prove that QRST is a parallelogram.

(vii) The line segments joining the mid points of opposite pairs of sides of any quadrilateral bisect each other.

4. <u>The following notes are included to indicate some of the definitions and properties of vectors that are useful in proving deductions</u>.

(i) If A and B are any two points (in 2- or 3-space), then $\overrightarrow{AB}$ denotes the "directed line segment AB", or (for the sake of brevity) "arrow AB".

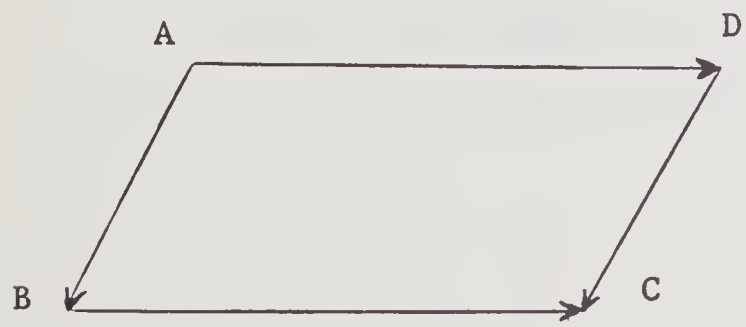
(ii) "$\overrightarrow{AB} = \overrightarrow{CD}$" means that $\overrightarrow{AB}$ and $\overrightarrow{CD}$ have the same length (magnitude) and direction, and are called equivalent arrows.

(iii) The set of all arrows of the same length and direction as $\overrightarrow{AB}$ is called an "equivalence class of arrows" or a "vector", denoted by "vector $\overrightarrow{AB}$" or more simply by a notation such as " $\vec{a}$ ".

"Vector $\overrightarrow{AB}$" means the vector of which (arrow) $\overrightarrow{AB}$ is one representative. Sometimes $\overrightarrow{AB}$ (rather than "vector $\overrightarrow{AB}$") is used to denote a vector - this usage is determined from the context.

If A = B, then $\overrightarrow{AB} = \vec{0}$ (the zero vector), which has length 0 and direction undefined.

(iv) "$\overrightarrow{AB} = \vec{a}$" means that $\overrightarrow{AB}$ is one of the arrows of (vector) $\vec{a}$; i.e. (arrow) $\overrightarrow{AB}$ is a representative of (vector) $\vec{a}$. Hence if $\overrightarrow{AB} = \overrightarrow{CD}$, then $\overrightarrow{CD} = \vec{a}$ (i.e. $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are representatives of the same vector $\vec{a}$). This remark is useful when dealing with parallelograms. For example, if ABCD is a parallelogram and we let

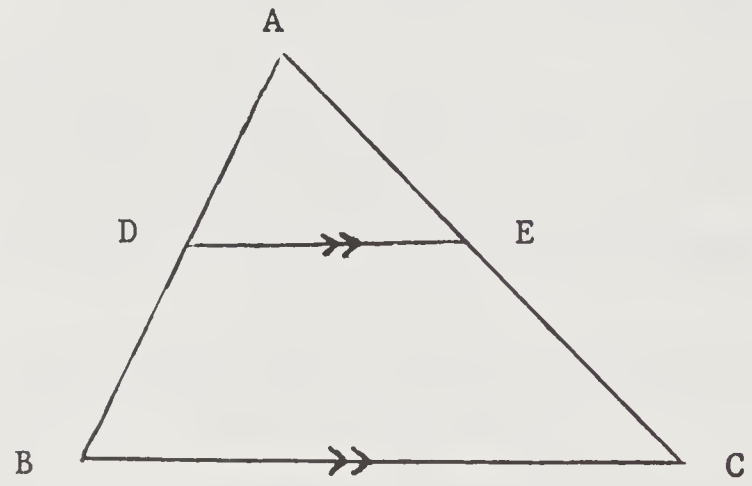


$\overrightarrow{AB} = \vec{a}$, then it follows $\overrightarrow{DC} = \vec{a}$, and if we let $\overrightarrow{AD} = \vec{b}$, then $\overrightarrow{BC} = \vec{b}$.

Also note that if $\overrightarrow{AB} = \vec{a}$, then $\overrightarrow{BA} = -\vec{a}$.

(v) If two arrows $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are such that AB $\parallel$ CD or A,B,C, and D are collinear, then $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are said to be collinear and either one is a scalar multiple of the other; for example $\overrightarrow{AB} = k\overrightarrow{CD}$ for some real k (if $k > 0$, then $\overrightarrow{AB}$ and $\overrightarrow{CD}$ have the same direction; if $k < 0$, then $\overrightarrow{AB}$ and $\overrightarrow{CD}$ have opposite directions; if $k = 0$, then $\overrightarrow{AB} = \vec{0}$). This is more generally applied to vectors; i.e. if AB $\parallel$ CD and $\overrightarrow{AB} = \vec{a}$, then $\overrightarrow{CD} = k\vec{a}$ for some real k ($\vec{a}$ and $k\vec{a}$ are said to be collinear). This is frequently used in describing geometric figures in which there are:

(a) parallel lines;



for example,

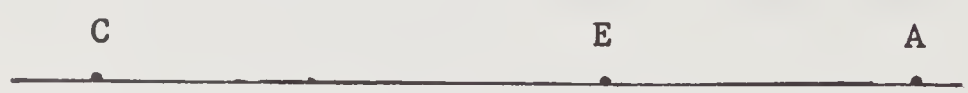
if DE $\parallel$ BC and

$\overrightarrow{BC} = \vec{a}$, then

$\overrightarrow{DE} = k\vec{a}$ for some $k \in \mathbb{R}$.

or

(b) two arrows in the same line; for example,



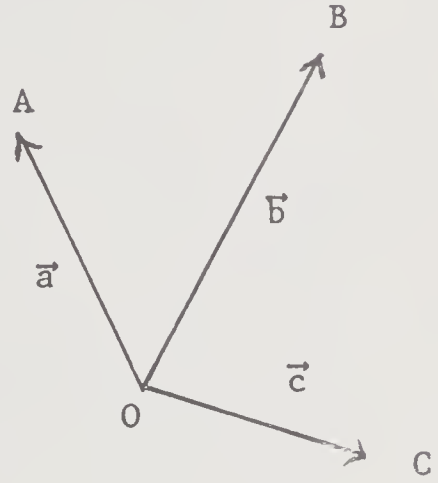
if $\overrightarrow{CA} = \vec{a}$, then $\overrightarrow{CE} = k\vec{a}$ for some $k \in \mathbb{R}$, and it follows that $\overrightarrow{EA} = (1 - k)\vec{a}$

(vi) If the scalar multiples of two vectors are equal, then either the vectors are collinear, or each of the scalar factors is zero. For example, if $p\vec{a} = q\vec{b}$ for some real p and q, then either $\vec{a}$ and $\vec{b}$ are collinear or $p = 0$ and $q = 0$. This latter result is most important in proving deductions, since if $\vec{a}$ and $\vec{b}$ are clearly not collinear and it is known that $p\vec{a} = q\vec{b}$ then it follows that $p = 0$ and $q = 0$. This technique has been used in 3. (i) (a) and (d) above.

5. In this section, a slightly different convention is introduced for denoting vectors, and a few deductions using this convention are examined.

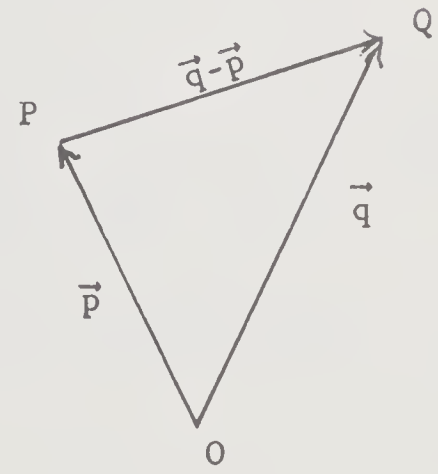
(i) <u>Convention</u>

Let O represent any point fixed in 2- or 3-space, then for points such as A, B, and C denote the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, and $\overrightarrow{OC}$ respectively by $\vec{a}$, $\vec{b}$, and $\vec{c}$ (as illustrated). Vectors denoted this way are called <u>position vectors</u>.



B
K 5
O 11
T 2
AA 9

(ii) By the vector addition law, it follows for any two points P and Q in space, that $\overrightarrow{PQ} = \vec{q} - \vec{p}$. <u>Hereafter we shall not show O in the diagram, nor the arrows $\overrightarrow{OP}$, $\overrightarrow{OQ}$ etc.</u>



(iii) Problem. If M is the mid-point of PQ, find an expression for $\vec{m}$ in terms of $\vec{p}$ and $\vec{q}$.

Solution: $\overrightarrow{PM} = \frac{1}{2}\overrightarrow{PQ}$,

Thus $\vec{m} - \vec{p} = \frac{1}{2}(\vec{q} - \vec{p})$;

and so $\vec{m} = \frac{1}{2}\vec{q} - \frac{1}{2}\vec{p} + \vec{p}$

$= \frac{1}{2}(\vec{p} + \vec{q})$.

This is immediately applicable in (iv), (v), and (vi) below.

(iv) Deduction

Prove that the line segment joining the mid-points of two sides of a triangle is parallel to the third side and equal to one-half of it.

Proof:

For ΔABC, with M and N mid-points of AB and AC respectively

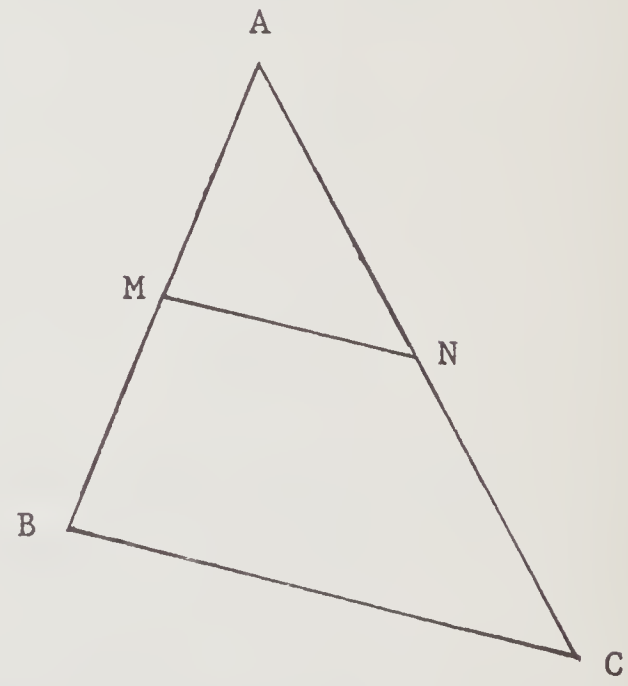
$m = \frac{1}{2}(\vec{a} + \vec{b})$ and $\vec{n} = \frac{1}{2}(\vec{a} + \vec{c})$.

Thus $\vec{n} - \vec{m} = \frac{1}{2}(\vec{a} + \vec{c}) - \frac{1}{2}(\vec{a} + \vec{b})$

$= \frac{1}{2}(\vec{c} - \vec{b})$;

i.e. $\overrightarrow{MN} = \frac{1}{2}\overrightarrow{BC}$,

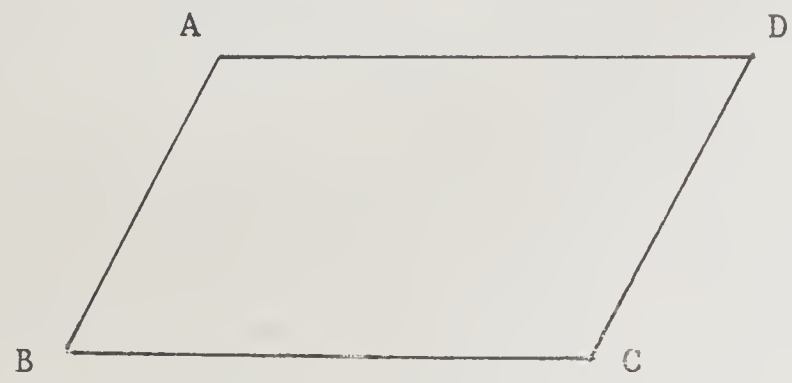
and so $MN = \frac{1}{2}BC$, $MN \parallel BC$.



(v) Deduction

Prove that the diagonals of a parallelogram bisect each other.

Proof: For parallelogram ABCD in space



$\overrightarrow{AB} = \overrightarrow{DC}$, and so $\vec{b} - \vec{a} = \vec{c} - \vec{d}$.

Hence $\vec{b} + \vec{d} = \vec{c} + \vec{a}$,

and $\frac{1}{2}(\vec{b} + \vec{d}) = \frac{1}{2}(\vec{c} + \vec{a})$.

This illustrates that the mid-point of BD is the same as the mid-point of CA; i.e., BD and CD bisect each other.

(vi) In any quadrilateral the bi-medians (line segments joining mid-points of opposite sides) and the line segment joining the mid-points of the diagonals are concurrent. (Hint: the point of concurrency is the centroid G of the vertices, and $\vec{g} = \frac{\vec{a} + \vec{b} + \vec{c} + \vec{d}}{4}$; this theorem is a special case of the "tetrahedron" theorem in Unit 4, Section 2., (v)). M (p. 199 #6)

The above discussion of mid-point of a line segment is a special case of the point of division of a line segment in a given ratio. This more general situation is discussed in Section 2. of Unit 4.

(vii) Most of the deductions listed in section 3. of this unit are amenable to "position vector" proofs.

D 13
E 10
F 10
G 9
K 6
O 10
P sec A,B
R 1-6
S 7
U 4
V 11,12
X geo.1
Y 4
Z 2
MM
NN
SS

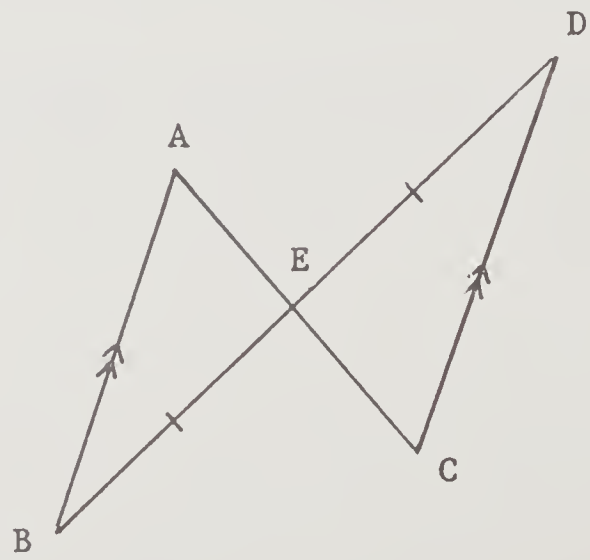
6. Rigid transformations (isometries) in the plane can be used to study many of the properties of geometry, which have been traditionally introduced by a Euclidean approach. This viewpoint is presented in the article, Congruence Geometry for Junior High School, by W. Sanders and J. Dennis, in the April 1968 issue of The Mathematics Teacher. Since students entering Grade 11 in Ontario have had prior introduction to some of the methods of Euclid, as well as to Transformations in Grade 10, some modifications in the suggested procedure will be needed. The authors first introduce experiences with the intuitive notions of "slides, turns, and flips" of models of geometric figures. These are then used to prepare the students for a later introduction of the one-to-one mapping (or functional) definitions of a translation, rotation, and a reflection. Applications of these latter definitions are then made to parallel lines, vertical angles, perpendicular lines, symmetry, triangles, quadrilaterals, and constructions. The properties so discovered can then be applied to the solution of deductions.

   The topic of transformations and its application to solving deductions is discussed in varying degrees of depth in many other articles and textbooks.

7. The following example illustrates the remarks made in section 2, on page 8. Three proofs are given, to illustrate the use of (i) transformations, (ii) Euclidean methods, and (iii) vectors as alternate approaches to the same deduction.

   Given: AB $\parallel$ CD and BE = ED.

   Prove: AB = CD.



SS

Comment: The proof below requires no accumulation of geometric properties - simply an intuitive acceptance of the following ideas.

Under a rotation of 180° about a point P (also called a half turn, Hp)

(i) $P' = P$ (X' is the image of X under Hp.)

(ii) a line through P maps into itself, so that the opposite rays from P are images of each other,

(iii) a line ℓ not through P maps into an image line ℓ' so that $\ell' \parallel \ell$

(iv) a rotation is rigid; that is, it preserves distances (and the measures of angles); i.e. $R'S' = RS$ for any points R and S in the plane.

Proof: Method 1, using transformations

Let $\Delta A'B'E'$ be the image of ΔABE under a rotation of 180° about E. $E' = E$, $B' = D$ and $A'B' \parallel AB$, and so A' lies in DC. A' also lies in EC.

Hence $A' = C$ and so $AB = A'B' = CD$.

The general ideas involved in the comment, and in the proof could be made more meaningful to the student by the use of an overhead projector and overlays, with a tracing of ΔABE on a separate overlay to illustrate the effect of the rotation.

Method 2, using a Euclidean approach

In Δ's ABE and CDE

	$BE = DE$	(why?)
	$\angle AEB = \angle CED$	(why?)
	$\angle ABE = \angle CDE$	(why?)
hence	$\Delta ABE \cong \Delta CDE$	(why?)
and	$AB = CD$	(why?)

Comment: This proof requires an accumulation of structure based on two theorems, a congruency postulate, and the definition of congruency.

Method 3, using vectors

Let vectors be denoted as in the diagram, where $p, k \in R$

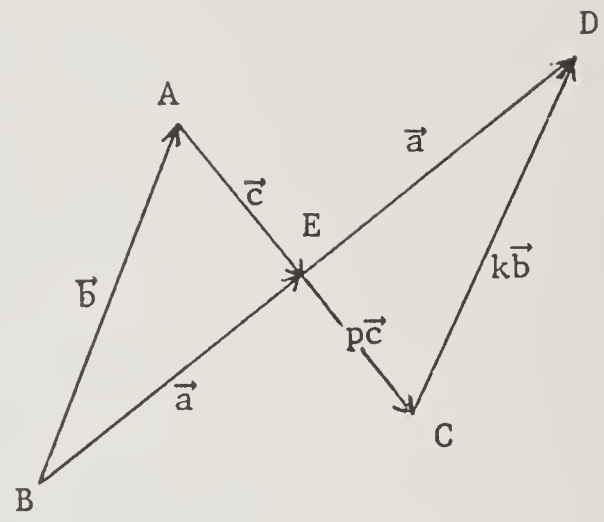
By vector addition, $\vec{a} = \vec{b} + \vec{c}$ and $\vec{a} = p\vec{c} + k\vec{b}$.

Thus $\vec{b} + \vec{c} = p\vec{c} + k\vec{b}$, and so $(1 - k)\vec{b} = (p - 1)\vec{c}$.

Since $\vec{b}$ and $\vec{c}$ are non-collinear $1 - k = 0$ and $p - 1 = 0$;

i.e. $k = 1 = p$

It follows that BA = CD (and incidentally that AE = EC).



Unit 3 AREAS AND THE PYTHAGOREAN RELATION

Suggestions for Adaptation

1. Although the students have had intuitive experiences with area and A 6
have used formulas in earlier grades to find the area of some of the B 6
regular plane figures, the properties of area as outlined in sections 1., C 6
3.,and 4. are likely new to them. The Pythagorean Theorem has been S 8
discussed and used in problems since Grade 8, and a proof of it has been
demonstrated in Grade 10. Other proofs of this theorem might be investi-
gated, along with Pythagorean Triples extended in the following manner: KK

- Suppose we extend the usual definition of a Pythagorean Triple (P.T.) to mean a partially ordered triple of non-negative numbers, say a, b, and c (denoted $[a, b, c]$) such that $c^2 = a^2 + b^2$. It is then simple to prove $[a, b, c] = [b, a, c]$, but $[a, b, c] \neq [a, c, b]$, etc., if these triples are Pythagorean.

- If we define $[a, b, c]\, k = [ka, kb, kc]$, where k is non-negative, it is trivial to prove that $[a, b, c]\, k$ is a P.T. if and only if $[a, b, c]$ is a P.T.

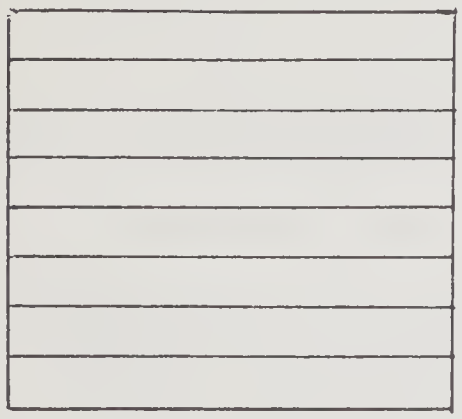
- This theorem may be used to abbreviate the solution of a right-triangle given any two of the sides, if the given sides have a highest common factor other than 1. For example, if the legs of a right-triangle are 105 and 147, then the P.T. representing the lengths of the sides is

$[5, 7, \sqrt{74}]$ 21. Thus the hypotenuse is $21\sqrt{74}$. This theorem may be readily applied to right-isosceles-triangles, and triangles with 30°, 60°, and 90° angles where the basic P.T.'s are $[1, 1, \sqrt{2}]$ and $[1, \sqrt{3}, 2]$ respectively.

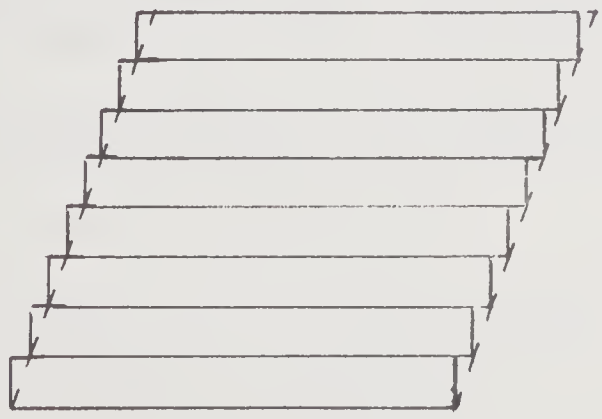
- Formulas for generating P.T.'s of integers might be investigated. For example, $[2mn, m^2 - n^2, m^2 + n^2]$, where $m,n \in I$ and $m > n > 0$, generates integral pythagorean triples. B 15

2. The Pythagorean Theorem should also be applied to three-dimensional figures; for example, to find the length of a diagonal of a right rectangular parallelopiped. Students should benefit from building three-dimensional models (for example, using boxes) to illustrate such line segments in space, and from making drawings to represent these three-dimensional objects. DD 12

3. Many of the ideas of 1. of Unit 3 in S.12A can be extended intuitively to volumes of prisms, and pyramids. The concept of constant area (volume) of a parallelogram (parallelopiped) of a given base and altitude can be illustrated by examining a stack of rectangles (rectangular blocks) as shown in diagrams (a) and (b), and noting in (b) that if the small triangles (prisms) on the left of each rectangle (block) were moved to the right end of the rectangle (block) a parallelogram (parallelopiped) of the same area (volume) would result. DD 12 Y 7



(a)



(b)

This demonstration can be extended to a stack of cards or a package of 1000 sheets of foolscap, and in so doing the notion of limit is implicitly introduced.

The above approach introduces intuitively the notion of a transformation called a "shear". Analytically speaking, a shear parallel to the x-axis may be defined by $(x,y) \rightarrow (x + ky, y)$ for some real k. Consequently the altitude of a figure with base on the x-axis remains constant under such a shear, and, since its base is unchanged, its area is constant. Shears can be demonstrated conveniently for two-dimensional figures by using a geo-board and figures made with elasticized line segments.

These investigations can be extended to include:

- triangles and polyhedrons on the same base and between the same parallels,
- Cavalieri's Theorem, A 13 S 8
- the inequalities occuring when figures have the same base and are <u>not</u> between the same parallels (different altitudes),
- areas and volumes of cylinders, cones, and spheres,
- drawing of nets, and the construction of models of regular polyhedra, cylinders, cones, etc.
- formulas for surface area and volume of various solids. $X_{arith.}^{2}$

Unit 4 RATIO AND PROPORTION

Suggestions for Adaptation

1. The concept of a two-termed ratio is introduced in both Grade 7 (I.12A, Section B, Unit 3) and Grade 8 (I.12B, Unit 5), and is applied inductively to similar triangles in Grade 9 (I.12C, Unit 6, Section 4(d)). Any reference to ratio in Grade 10 is incidental. In Section 1 of Unit 4 in S.12A, a general definition of ratio is introduced as an ordered n-tuple of numbers not all zero. This should be used to clarify and consolidate the student's previous understanding of ratio. The student should now recognize that (i) the ratio $a:b$ where $a, b \in R$ and the number $\frac{a}{b}$ where $a, b \in R$, $b \neq 0$ do not represent the same idea, but that for each ratio $a:b$ if $b \neq 0$, there corresponds a real number $\frac{a}{b}$, and (ii) for a ratio of 3 or more terms there is not a corresponding real number. A 7 B 7 C 7

2. Division of a line segment in a given ratio, using vector methods

After the student is familiar with the basic notions of division of a line segment in a given ratio, the following ideas could be investigated by using a further application of position vectors (as outlined in Unit 2, section 5. of this statement). T 2

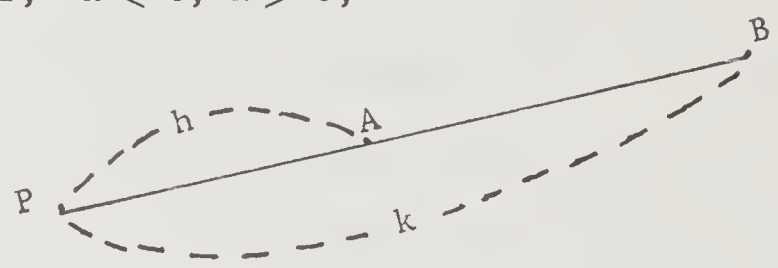
The vector $\vec{m} = \frac{1}{2}(\vec{a} + \vec{b})$ where M is the mid-point of AB (see Unit 2, section 5, (iii)) is a special case of the vector $\vec{p}$ where P divides segment AB in a given ratio, as discussed below. This result can be used to investigate many interesting geometric properties.

(i) Theorem If P divides line segment AB in the ratio h:k (where $h \neq 0$, $k \neq 0$, $h + k \neq 0$), then $\vec{p} = \frac{h\vec{b} + k\vec{a}}{h + k}$. AA 9 K 5

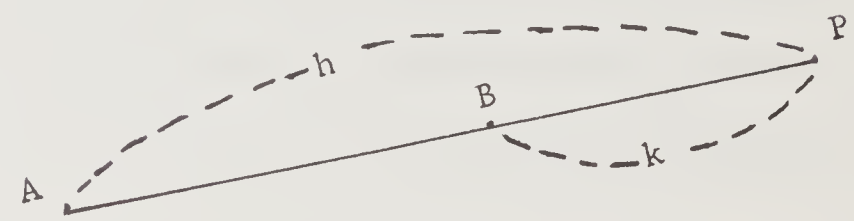
Proof: Three cases exist. Assume $\overrightarrow{AB}$ indicates a positive direction, then

(a) if $P \in \overline{AB}$, $h > 0$, $k > 0$;

(b) if $P \in$ ray BA and $P \notin \overline{AB}$, $h < 0$, $k > 0$;



(c) if $P \in$ ray AB and $P \notin \overline{AB}$, $h > 0$, $k < 0$.



For each of these cases:

$$\frac{AP}{PB} = \frac{h}{k},$$

(from the context, AP represents a directed distance from A to P)

i.e. $AP = \frac{h}{k} PB;$

and so $\overrightarrow{AP} = \frac{h}{k} \overrightarrow{PB}.$

Thus $\vec{p} - \vec{a} = \frac{h}{k} (\vec{b} - \vec{p})$,

or $(k + h)\vec{p} = h\vec{b} + k\vec{a}$;

and so $\vec{p} = \frac{h\vec{b} + k\vec{a}}{h + k}$.

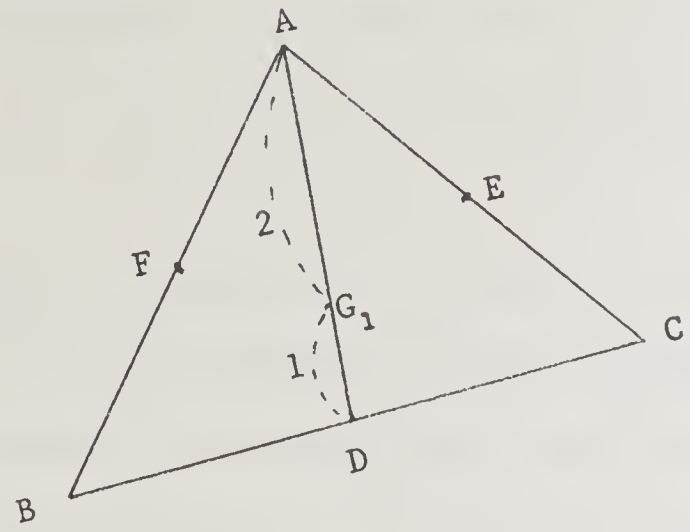
For any point O, if P divides segment AB in the ratio h:k,

vector $\overrightarrow{OP} = \vec{p} = \frac{h\vec{b} + k\vec{a}}{h + k}$.

This theorem may be used to derive a simpler proof to the following deduction, which was first introduced in Unit 2.

(ii) Deduction The medians of a triangle are concurrent, and trisect each other. K 5 T 2 AA 9

Proof:



D, E, and F are the mid-points of BC, CA, and AB of ΔABC.

Let G_1, G_2, G_3 be the points dividing AD, BE, and CF in the ratio 2:1.

Hence $\vec{g}_1 = \frac{2\vec{d} + \vec{a}}{2 + 1} = \frac{2 \cdot \frac{1}{2}(\vec{b} + \vec{c}) + \vec{a}}{3} = \frac{\vec{a} + \vec{b} + \vec{c}}{3}$.

Similarly $\vec{g}_2 = \frac{\vec{a} + \vec{b} + \vec{c}}{3}$, and $\vec{g}_3 = \frac{\vec{a} + \vec{b} + \vec{c}}{3}$.

This shows that $G_1 = G_2 = G_3$, and so the medians of a triangle are concurrent at G, the point of trisection of each median farthest from the vertex.

G is called the centroid of the vertices of the triangle. M 13.6

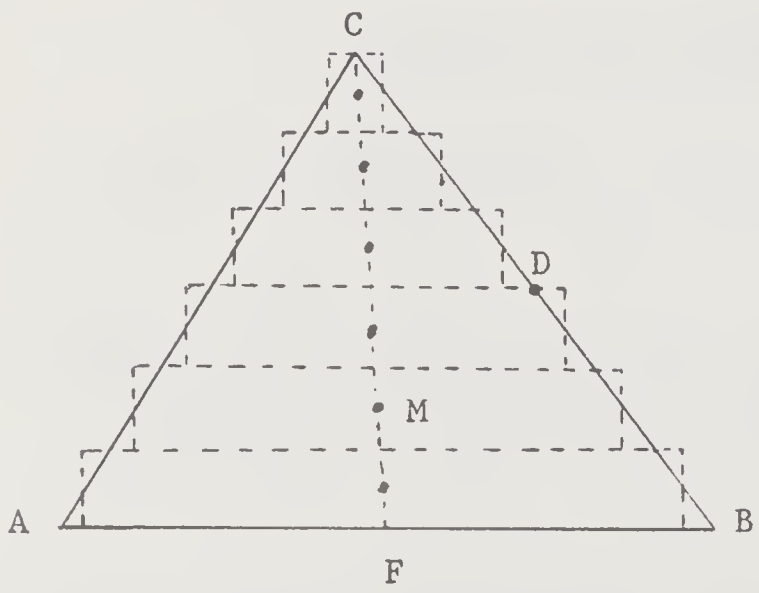
If we consider a system of equal unit weights positioned at A, B, and C, then G is also called the centre of gravity of this system. (The given system is equivalent to a 2 unit weight at D with a 1 unit weight at A, which in turn is equivalent to a 3 unit weight at G where G divides AD in the ratio 2:1.)

(iii) In general, the <u>centroid of a polygonal region</u> (often called the centroid of a polygon - in the strict sense, mistakenly so) is <u>not</u> the same point as the <u>centroid of the vertices of the polygon</u>, nor of the polygon (polygonal path). For a triangle, however, the centroid of the triangular region is the same point as the centroid of its vertices (although not as the centroid of the triangle, in general). This is discussed below.

<u>Note</u>:

The notion of centre of gravity (centroid) of a triangular lamina (region) might now be <u>intuitively</u> discussed, to show that it is the same point as the centroid of the vertices of the triangle. This can be done by replacing the lamina by n rectangular laminas, and considering the centre of gravity of each of these as its mid-point M (balance a metre stick at its mid-point), then discussing the effect of 2n, 4n, ... such rectangular laminas, until the centre of gravity of the triangular lamina can be assumed to lie in the median CF. This procedure may be used to show that the centre of gravity is also in one of the other medians, say AD, and is thus at the point of intersection of any two medians.

M 13.6



This shows intuitively that the centre of gravity (centroid) of a triangular lamina (region) is the same point as the centroid of the vertices of the triangle.

The above may be reinforced by geometrically locating the centroid of the vertices of a triangular lamina and showing that indeed this is the point of balance of the lamina.

In Unit 2, section 5, (vi), the centroid of the vertices of a quadrilateral is discussed. In general this point is not the centroid of the quadrilateral region. This may be demonstrated by the method of the previous paragraph, and opens up an investigation of determining special polygonal regions for which the centroids of the vertices and of the region coincide (consider for example, a square, a rectangle, regular n-gons, etc.). This may be extended, after (v) below, for solid figures.

M (p.216, #3,4)

(iv) Deduction

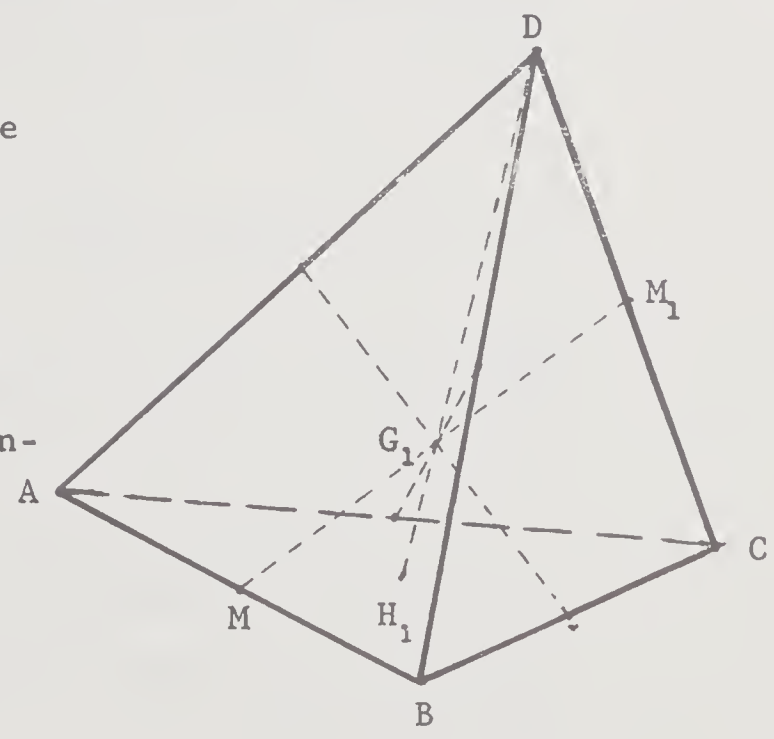
If D, E, F are mid-points of the sides of ΔABC, show that the centroid of ΔDEF is the centroid of ΔABC, the medians of ΔDEF are contained in the medians of ΔABC, and the medians of ΔABC are bisected by the sides of ΔDEF. Can this idea be extended for ΔDEF, etc.?

(v) Deduction

In a tetrahedron ABCD, the three bi-medians (line segments joining the mid-points of opposite edges) bisect each other, and the four medians (line segments from a vertex to the centroid of its opposite face) divide each other in the ratio 3:1, and all are concurrent at the point G, where

$$\vec{g} = \frac{\vec{a} + \vec{b} + \vec{c} + \vec{d}}{4}$$

T 2
AA 9



Solution:

Let G_1 be the mid-point of bi-median MM_1.

Thus $\vec{g}_1 = \frac{\vec{m} + \vec{m}_1}{2} = \frac{1}{2}\left(\frac{\vec{a} + \vec{b}}{2} + \frac{\vec{c} + \vec{d}}{2}\right)$

$= \frac{1}{4}(\vec{a} + \vec{b} + \vec{c} + \vec{d})$.

If G_2 and G_3 are the mid-points of the other bi-medians it can be shown that $\vec{g}_2 = \vec{g}_3 = \frac{1}{4}(\vec{a} + \vec{b} + \vec{c} + \vec{d})$.

Let H_1 be the centroid of ΔABC, and G_4 divide DH_1 3:1, then

$\vec{g}_4 = \frac{3\vec{h}_1 + \vec{d}}{3 + 1}$ by (i)

$= \frac{1}{4}(3 \cdot \frac{1}{3}(\vec{a} + \vec{b} + \vec{c}) + \vec{d})$ by (ii)

$= \frac{1}{4}(\vec{a} + \vec{b} + \vec{c} + \vec{d})$.

If G_5, G_6, G_7 are the points dividing the other segments from a vertex to the centroid of its opposite face 3:1, then it can be shown $\vec{g}_5 = \vec{g}_6 = \vec{g}_7 = \frac{1}{4}(\vec{a} + \vec{b} + \vec{c} + \vec{d})$.

Thus the bi-medians and medians are concurrent at G, which is called the centroid of the vertices of the tetrahedron.

The effect on the position of G if D is positioned closer to the plane of ΔABC could be discussed. If D is in the plane of ΔABC the tetrahedron degenerates into a triangle, or a quadrilateral; however the general result proved above for the tetrahedron still holds. (See Unit 2, section 5, (vi).)

Note:
An argument similar to that given in the Note of (iii), can be advanced to show that the centroid, or centre of gravity, of a solid tetrahedron is also the centroid of its vertices. This would involve examining the centroids of n triangular sections parallel to one face, and repeating this procedure for at least one other face.

In general, related to polyhedrons, there are four distinct centroids to consider: of the (i) vertices, (ii) union of the edges, (iii) union of surfaces, and (iv) solids. These do not usually coincide. Students might be interested in investigating solids for which they do.

(vi) A tetrahedron is a triangular pyramid. If we consider now pyramids with bases a quadrilateral, pentagon, hexagon, etc. in each case special sets of concurrent segments exist. For five vertices, A,B,C,D, and E, the segments (i) from a vertex to the centroid of the other four divide each other 4:1, and (ii) from the centroid of two vertices to the centroid of the other three vertices divide each other 3:2, and all of these are concurrent at a point with position vector $\vec{g} = \frac{\vec{a} + \vec{b} + \vec{c} + \vec{d} + \vec{e}}{5}$.

For six vertices, A,B,C,D,E, and F, the segments (i) from a vertex to the centroid of the other five vertices divide each other 5:1, (ii) from the centroid of two vertices to the centroid of the other four divide each other 2:1, and (iii) from the centroid of three vertices to the centroid of the other three bisect each other, and all of these are concurrent at a point G, with position vector $\vec{g} = \frac{\vec{a} + \vec{b} + \vec{c} + \vec{d} + \vec{e} + \vec{f}}{6}$.

The above situation generalizes for the centroid G of the n vertices, A_n, of a pyramid (or polyhedron) as being determined by $\vec{g} = \frac{\vec{a}_1 + \vec{a}_2 + \ldots + \vec{a}_n}{n}$.

3. After the students are familiar with the introduction to ratio and proportion, as outlined in Section 1, it might be profitable to discuss meaningful problems from science involving variation. It will be beneficial to all concerned if the science and mathematics departments meet to discuss suitable problems, and to correlate the approaches being used to solve these problems in each of the subject areas. Y 9,10 $X_{alg.}2$

The following notes (a), (b), and (c), are included to indicate some possible ideas for discussion.

(a) Students could investigate the <u>relation</u> between the <u>dimensions</u> of some regular two- and three-dimensional figures (squares, rectangles, cubes, right rectangular parallelopiped, cylinders, cones, etc.) and their <u>areas</u>, or <u>surface areas</u> and <u>volumes</u>, respectively. By examining various models of two- and three-dimensional objects and from student developed tables (such as the one indicated below), they may observe that the surface area of a solid increases fourfold, and its volume eightfold when its linear dimensions are doubled, etc. More generally they might conjecture that if each of the dimensions of a regular solid is changed by the same factor n, then the surface area and the volume of the solid <u>vary directly</u> as the square, and the cube, respectively of n. The above ideas may be expressed mathematically using equations (formulas) developed by the method in (c) below, and the relations may be illustrated by drawing their graphs. DD 13,14

Right Rectangular Parallelopiped

Dimensions (linear units) length	width	height	Surface Area (square units)	Volume (cubic units)
4	3	2	52	24
8	6	4	208	192
12	9	6	468	648
⋮	⋮	⋮	⋮	⋮
a	b	c	$2(ab + ac + bc)$	abc
2a	2b	2c	$4[2(ab + ac + bc)]$	$2^3(abc)$
3a	3b	3c	$9[2(ab + ac + bc)]$	$3^3(abc)$
⋮	⋮	⋮	⋮	⋮
na	nb	nc	$n^2[2(ab + ac + bc)]$	$n^3(abc)$

(b) The general principle discovered in (a) may be related to experimental situations in Physics, Chemistry, and Technology, as illustrated by the following examples: HH 9 II 7

(i) Rate of cooling

The rate of cooling of a solid varies directly as the measure of the surface area of the solid.

Since the volume of an object (and hence its mass) varies as the "cube" of its dimensions, (cube of the factor by which each dimension increases), and the surface area varies as the "square" of its dimensions, the larger of two similar solids of the same material will cool at a slower rate from a given initial temperature.

(ii) Air resistance

The resistance offered to a falling object by the air (or other medium such as water or oil) is directly proportional to the surface area exposed in the direction of the fall, and the gravitational force acting in the direction of free fall is proportional to the mass of the object.

Doubling the dimensions of an object will increase the air resistance fourfold but will increase the gravitational force acting in the direction of fall eightfold. Thus an increase in the dimensions of an object with a corresponding increase in mass will cause it to fall at a greater rate in the presence of air resistance.

(c) In problems dealing with variation, a relation is frequently described by an equation (rule of correspondence) of the form $y = f(x)$, or as a continued "proportion" of the form $\frac{y_1}{x_1} = \frac{y_2}{x_2} = \frac{y_3}{x_3} = \ldots = k$.

The following example is intended to illustrate some of the terminology and notations commonly used in science and in mathematics for such situations. II 7

Example. It can be noted, from experimentation and by analysis of the data obtained, that the magnitude (F) of the force of a wind acting on a surface increases with the magnitude (s) of the wind velocity, in such a way that $\frac{F}{s^2}$ is constant for all pairs of values of s and F.

The mathematician describes this situation by saying "F is a function (say g) of s", and denotes this function by:

$$\left.\begin{array}{ll} & F = g(s), \text{ where } g(s) = ks^2, \\ \text{or} & F = ks^2, \\ \text{or} & g: s \rightarrow ks^2, \\ \text{or} & g = \{(s,F) \mid F = ks^2\}, \end{array}\right\} \quad \begin{array}{l} \text{for some real } k, \\ \text{where } s \geq 0,\ s \in R. \end{array}$$

It is still common practice in science to describe the above situation by saying

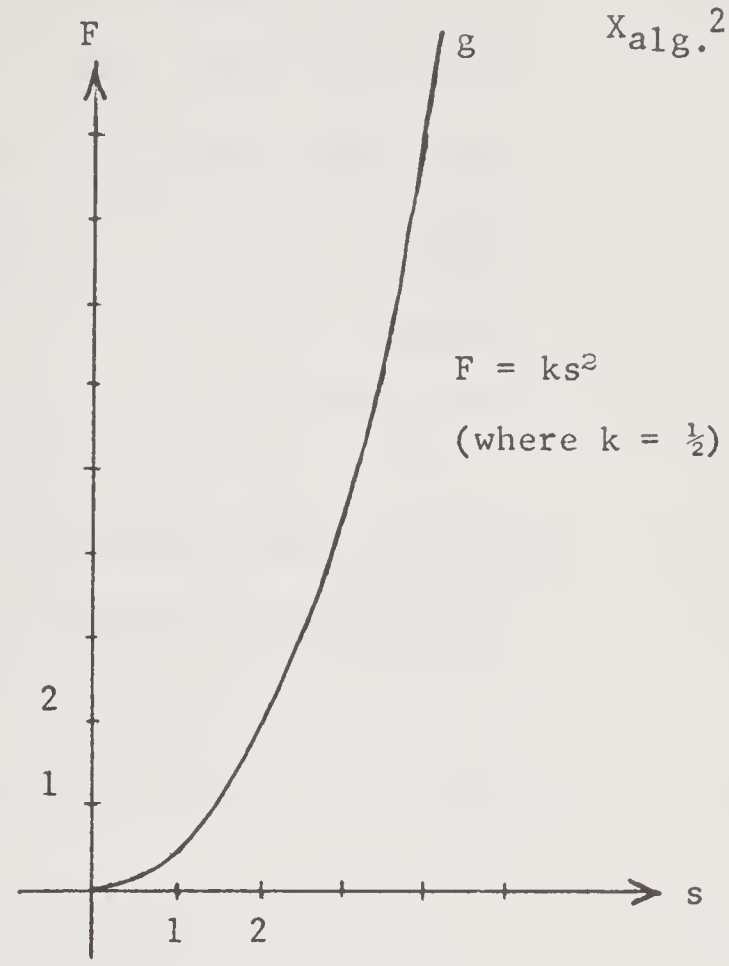
"F varies directly as (is proportional to) the square of s", and to denote it by

(1) $$F \propto s^2$$

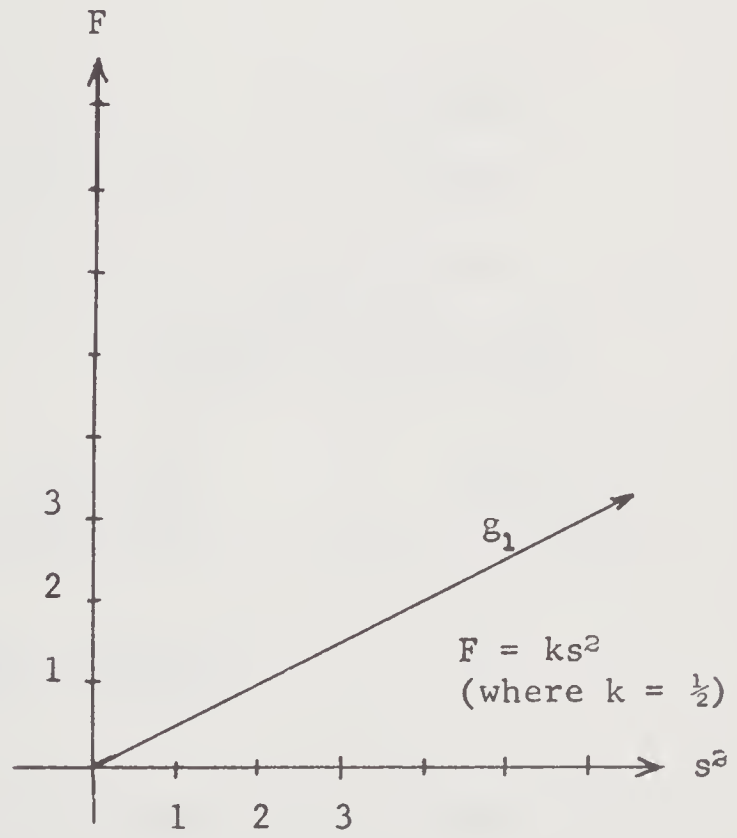
where "$\propto$" represents "varies as" or "is proportional to".

In the above discussion, s is called the independent variable, and F the dependent variable.

The graph of g is parabolic in form, and shows that $\frac{F}{s}$ is <u>not</u> constant and that F does <u>not</u> vary directly as s; i.e. the rate of change of F with respect to s is not constant, and that F changes more rapidly than s, when $s > 1$.

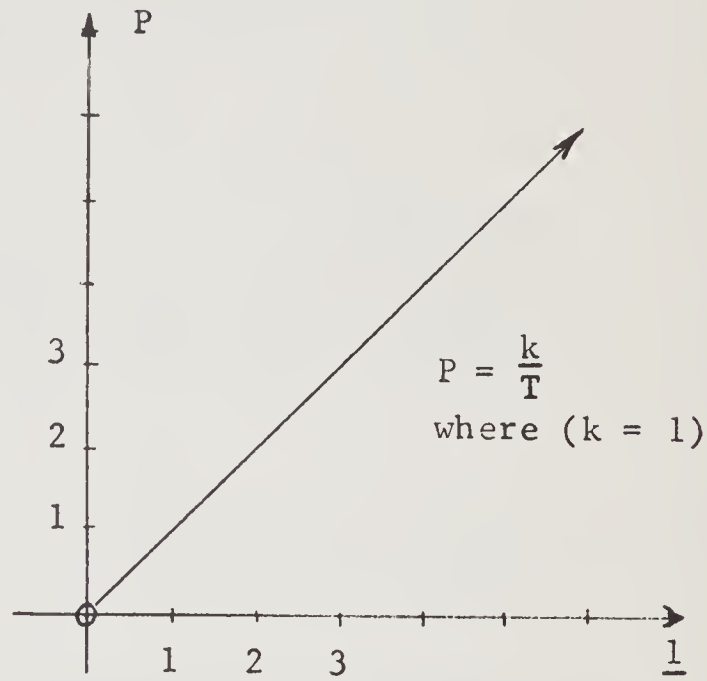


If, however, we consider s^2 as the independent variable, the graph of $g_1 = \{(s^2,F) \mid F = ks^2,\ s \geqq 0\}$ is a ray from the origin, of slope k. This illustrates that F <u>varies directly</u> as s^2; and that the rate of change of F with respect to s^2 is constant.



This latter graph is particularly useful for interpolation and extrapolation of data. This technique of "re-labelling" the axis of the independent variable so that the graph is part of a straight line is frequently used by the scientist. For example, in situations involving <u>inverse</u> variation, such as $P \propto \frac{k}{T}$ and $T > 0$, the relation may be described by saying that

P varies <u>directly</u> as the reciprocal (multiplicative inverse) of T, and by labelling the axes $\frac{1}{T}$ and P the graph is an "open ray" from the origin.



S p.45

The equation $F = ks^2$ (or $P = \frac{k}{T}$) is the rule of correspondence of the function, and by the substitution of a pair of values for s and F (or T and P) the value of k is determined for the particular function. It follows then that

(2) $\frac{F_1}{s_1^2} = \frac{F_2}{s_2^2}$ for the direct variation example, and

(3) $P_1 T_1 = P_2 T_2$ for the inverse variation example.

4. The notions of similar geometric figures can be effectively introduced by means of dilatations in the plane (a transformation by which a geometric figure is mapped into an image having the same shape as the object). These ideas extend easily into corresponding situations for 3-space.

O 10
P sec C
R 7,8
S 5,7
U 4
Y 8

If the concept of similar figures is introduced by means of dilatations, then by accepting intuitively a few basic properties of dilatations (dealing with invariance of angle size, magnification of the length of line segments, lines through the centre of dilatation, collinear points, and parallel object and image lines) many properties of similar figures can be proved quite simply.

5. The basic notions of geometric vectors (as described in 3. to 5. of Unit 2 and 2. of Unit 4 in this statement) can be used to develop proofs for many of the theorems and deductions of Unit 4 of S.12A.

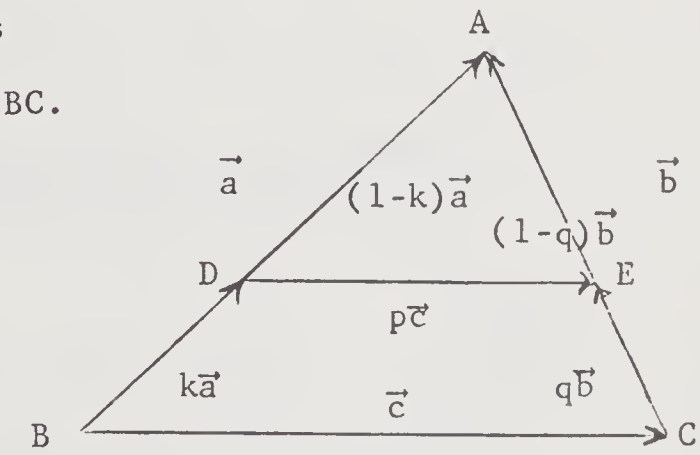
This is illustrated below in (i) and (ii).

(i) Theorem (b), Page 7 of S.12A
A straight line intersecting two sides of a triangle is parallel to the third side if and only if it divides the two sides in the same ratio.

Part 1

Given: $\triangle ABC$, and points D and E in sides AB and AC such that DE is parallel to BC.

Required to prove: $\frac{AD}{DB} = \frac{AE}{EC}$.



Proof: Vectors may be labelled as indicated in the diagram. (It is sufficient to show that $k = q$, where $k, q, p \in R$.)

$k\vec{a} + p\vec{c} = c + q\vec{b}$, and $\vec{a} = \vec{c} + \vec{b}$. } vector addition

By replacement, $k(\vec{c} + \vec{b}) + p\vec{c} = \vec{c} + q\vec{b}$,

i.e. $(k + p - 1)\vec{c} = (q - k)\vec{b}$.

Since $\vec{b}$ and $\vec{c}$ are not collinear, then

$k + p - 1 = 0$ and $q - k = 0$;

i.e. $q = k$ and $p = 1 - k$.

The required result follows from $q = k$.

Note: Since $p = 1 - k$ we have

$$\frac{AD}{AB} = \frac{AE}{AC} = \frac{DE}{BC} = \frac{1-k}{1}.$$

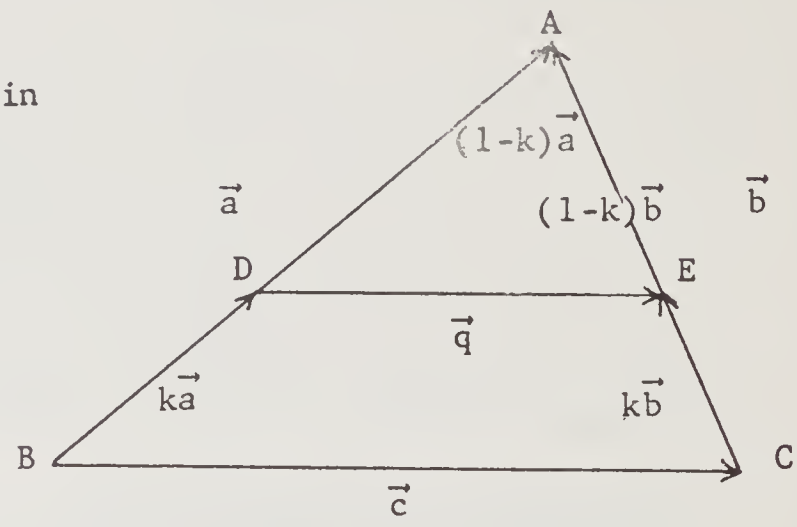
This information is a bonus – it was not anticipated. Frequently, when proving deductions using vectors we obtain additional information from the solution. Students should learn to expect this when working with vectors, and be on the look out for it.

Part 2

Given: ΔABC and points D and E in AB and AC respectively such that

$\frac{AD}{DB} = \frac{AE}{EC}$.

Required to prove: DE $\parallel$ BC.



Proof: Vectors may be labelled as shown in the diagram, where $k \in R$.

$$\left.\begin{aligned} k\vec{a} + \vec{q} &= \vec{c} + k\vec{b}, \\ \text{and} \quad \vec{a} &= \vec{c} + \vec{b}. \end{aligned}\right\} \quad \text{vector addition}$$

By replacement $k(\vec{c} + \vec{b}) + \vec{q} = \vec{c} + k\vec{b}$;

i.e. $\vec{q} = (1 - k)\vec{c}$.

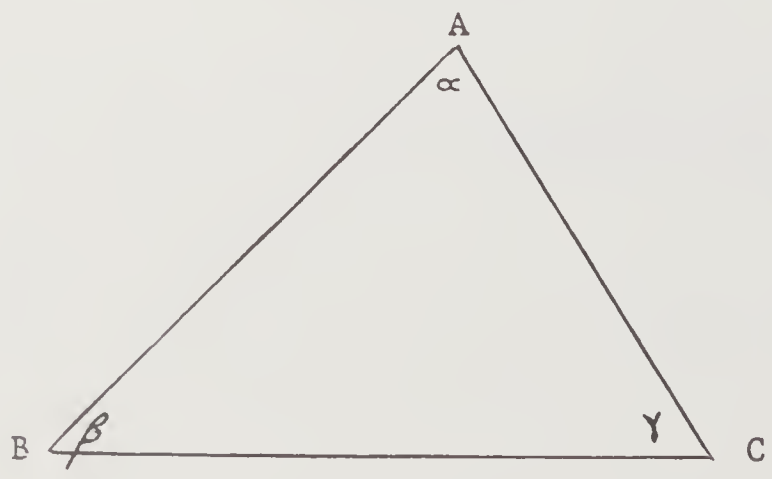
Thus $\vec{q}$ is a scalar multiple of $\vec{c}$, (since $1-k \in R$), and so DE $\parallel$ BC.

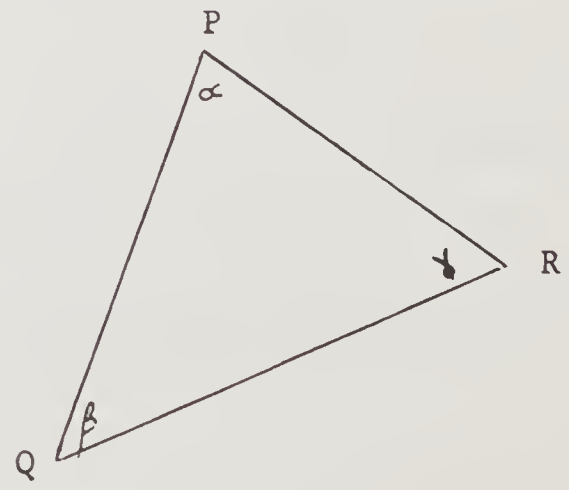
(ii) Theorem (c), Page 7 of S.12A
Two triangles are equiangular if and only if their corresponding sides are proportional.

Part 1

Given: Δ's ABC and PQR in which $\angle A = \angle P$, $\angle B = \angle Q$, and $\angle C = \angle R$.

Required to prove: $\frac{PQ}{AB} = \frac{PR}{AC} = \frac{QR}{BC}$.





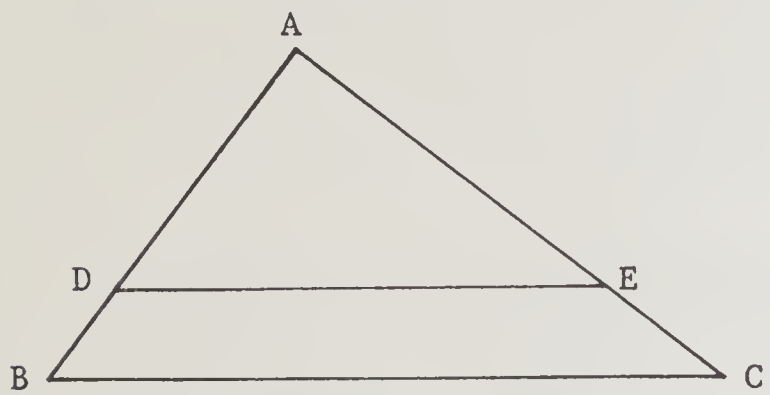
Discussion

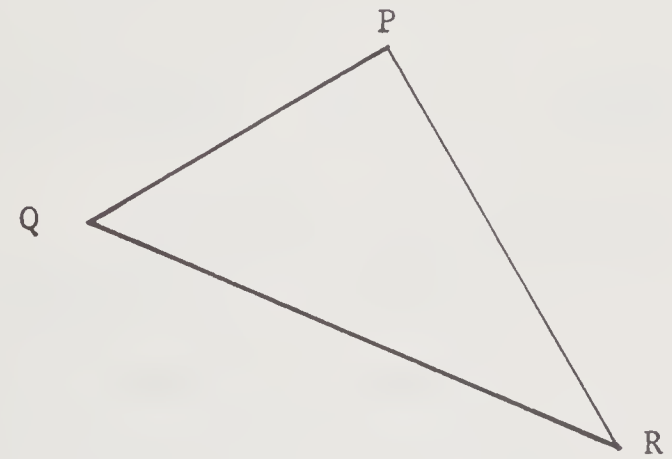
By means of a suitable congruency transformation (some combination of a translation, reflection, and a rotation, and thus rigid) P is mapped onto A and PQ is mapped into the ray AB, and (because the transformation is rigid and $\angle QPR = \angle BAC$) PR is mapped into the ray AC. Let $Q'R'$ be the image of QR. The observation that $Q'R'$ is parallel to BC and the Note of Part 1. of (i) lead to the required result.

Part 2

Given: Δ's PQR and ABC in which $\frac{PQ}{AB} = \frac{PR}{AC} = \frac{QR}{BC}$.

Required to prove: ΔPQR equiangular to ΔABC.





Proof:

$\frac{PQ}{AB} = \frac{PR}{AC} = \frac{QR}{BC} = \frac{k}{1}$, for some $k \in R$. (1)

Points D and E exist in rays AB and AC such that AD = PQ and AE = PR.

Hence $\frac{AD}{AB} = \frac{AE}{AC} = \frac{k}{1}$, (replacement in (1))

and so $\frac{AD}{DB} = \left(= \frac{k}{1-k}\right) = \frac{AE}{EC}$,

and DE $\parallel$ BC. (Part 2. of (i))

Thus $\frac{AD}{AB} = \frac{AE}{AC} = \frac{DE}{BC} = \frac{k}{1}$. (Note of Part 1 of (i))

Thus $DE = k \cdot BC$.

But from (1) $QR = k \cdot BC$, and so $DE = QR$ and $\Delta ADE \cong \Delta PQR$.

Since DE $\parallel$ BC, it may be shown that ΔADE (and consequently triangle PQR) is equiangular to ΔABC.

(iii) Deduction
Four non-collinear points O, P, Q, and R are given. If $\overrightarrow{OP} = \vec{a}$, $\overrightarrow{OQ} = \vec{b}$, $\overrightarrow{OR} = \vec{c}$ and $\vec{b} = m\vec{a} + n\vec{c}$ where $m,n \in R$ and $m + n = 1$, prove that P, Q, and R are collinear.

6. Students might be encouraged to investigate the properties of similar solid figures as extensions of the properties of similar plane figures. This extension can be introduced through dilatations about a centre of dilatation in 3-space, and mapping of any point P in a solid figure into an image P′ so that $\frac{OP'}{OP} = \frac{k}{1}$, for the magnification factor k.

Unit 5 TRIGONOMETRY

Suggestions for Adaptation

1. If this unit is discussed after Units 6 and 7, it can be introduced as a further illustration of the function concept. (At the present time in the S.12 sequence of courses, functions are not discussed until S.12B; however, students who have completed the I.12 sequence will have discussed functions in Grade 10.)

A 8
B 9
C 8
$X_{trig.}$ 1
Y 12
Z 4
DD 7,10
EE 4
UU

In recent years many teachers have had success in introducing the trigonometric functions by a discovery approach. To illustrate: the class is divided into groups and each group draws on squared paper a large circle (or circles) with centre the origin and any radius. Radii are then drawn at 10° intervals for angles from 0° to 360°. The students then record the ordinate y and abscissa x for the end point of each radius, and tabulate values for y/r, x/r, and y/x. Graphs are then drawn to illustrate the relation between the measure of the angle and each of y/r, x/r, y/x. These graphs are then discussed to indicate some of

the properties of the relations (now recognized as functions). At this stage the definitions of the primary functions may be introduced.

Some variations in the above procedure are:

(i) to draw two or more such concentric circles (in order to discover that the function values are independent of r),

(ii) to introduce angular measure in radians, and

(iii) to find the relation between the arc length (counterclockwise from (1, 0) on a unit circle) and y, then x, then y/x, where (x, y) is on the circle.

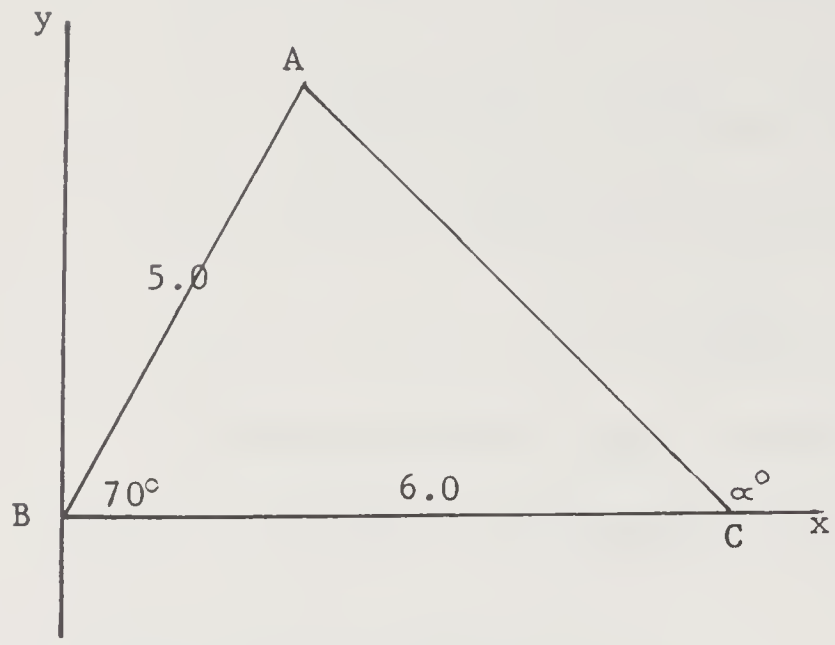
It is suggested that $\sin \theta^\circ$, $\cos \theta^\circ$, etc., be referred to as the trigonometric function values (i.e. value of $\sin 30^\circ$, etc.) since they represent real numbers (not ratios).

Commercially prepared apparatus is available to illustrate the trigonometric functions, known by such names as "Trig Tracker" or "Trig Aide".

Many of the properties of the trigonometric functions are evident from their graphs, such as the complementary relations (co-functions), use of tables for angles beyond 90°, maximum and minimum values, range of the functions, etc. It is recommended that students prepare posters of these graphs for display in the classroom, and that students be encouraged to use these graphs, or simple sketches of them, to intuitively reinforce their thinking.

2. The supplementary topics in Section 8 of S.12A might be examined, if time permits.

3. Solution of oblique triangles could also be examined by means of analytic vectors. For example, a solution is given for ΔABC.



$$\vec{CA} = \vec{CB} + \vec{BA}$$

$$= (-6.0,0) + (5.0\cos 70^\circ,\ 5.0\sin 70^\circ)$$

$$= (-6.0 + 5.0\cos 70^\circ,\ 5.0\sin 70^\circ)$$

$$= (-4.39,\ 4.70)$$

$$|CA| = \sqrt{(4.39)^2 + (4.70)^2}$$

$$\doteq 6.43$$

$$\tan \alpha^\circ \doteq \frac{4.70}{4.39} = -1.07$$

$$\alpha \doteq 133$$

Thus $\angle ACB = 47^\circ$, $\angle BAC = 63^\circ$.

The calculations required above should be done with a 10-key calculator or a slide rule.

Further references on this use of analytic vectors will be provided in the suggestions for teachers in the new Grade 11 and 12 Four-year course of study.

4. The inner product (also called dot product, or scalar product) could now be discussed, and applied to problems from geometry involving perpendicularity and calculation of the angle size between two lines.

K 5
O 11
T 4
BB 13
DD 18

Unit 6 LINEAR EQUATIONS AND INEQUATIONS

Suggestions for Adaptation

1. Many of the concepts in this unit are familiar to the student from his work with I.12D, and in earlier courses. Full advantage should be taken of this experience to abbreviate the treatment of this unit.

A 9
B 12
C 9,10
Z 5
II 8,9

2. The student is familiar with the concepts of relation and function from I.12D. These ideas should be kept alive and possibly extended to other than linear examples, in order that the student recognizes that not all equations determine linear graphs. In the statement of S.12A no reference is made to these concepts, but their presence is implicit in the course, and reference to them occurs in some of the present Grade 11 texts.

3. If time permits 5. (b) and (c) in Unit 6 of S.12A should be covered, as they are of value to the student in preparing him for topics in future courses; 5.(a) is interesting historically and as a problem-solving method.

4. The students could also investigate:

   (i) the solution of a system of two linear equations by using matrices,

   (ii) linear programming.

   These are included as supplementary topics in I.12D and the new Grade 10 texts contain material on both of them.

C 9
D 6,7
E 5
F 7
G 4
O 7
U 3
BB 21
EE 2

Unit 7 ANALYTIC GEOMETRY

Suggestions for Adaptation

1. In Unit 2, Section 4. of I.12D the student has been introduced to the meaning of slope, the recognition of the roles of m and b in equations of the form $y = mx + b$, and how to use $y = mx + b$ to find the equation of a line when given:

A 10,11
B 13,14
C 11
D 8
E 6
F 5
G 6

   (i) a value for b and the co-ordinates of a point in the line

   (ii) the co-ordinates of two points in the line.

   The emphasis in Unit 7 of S.12A is now on developing the equation of a straight line, given certain geometric conditions. The more intuitive experiences of the students in Grade 10 have probably prepared them for a richer understanding of the topics in this unit of S.12A.

2. Some three-dimensional work might be discussed, such as: the length of a line segment in three-space; the equation of a plane and its graph; the intersection of two planes, the intersection of three planes (both illustrated graphically).

3. When discussing slope, reference might be made to the rate of change of y with respect to x for equal changes in x, and this discussed for linear and non-linear examples, such as graphs of $y = x$ and $y = x^2$. This might be extended to include the numerical calculation of the slope of a tangent at a point (using a desk calculator).

F 6

4. It is advisable to develop the formula for the length of a perpendicular between a point and a line, i.e. $\frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$, and to give the students experience with using this formula in problems. The other supplementary topics might be discussed, if time permits.

5. Some of the properties of geometry might be discussed, using generalized analytic proofs, such as: the concurrency of altitudes, right bisectors, and medians; the medians trisect each other; parallelogram properties; rhombus properties.

REFERENCE BOOKS

The following list is extensive, but is by no means complete. The same topics are covered in many of these books from a variety of viewpoints; some of the books are suitable only as teacher references.

For the addresses of publishers not listed below, see Circular 14, 1968, pages 70 and 71.

Textbooks designed for Curriculum S.12A, Grade 11

A. Baxter et al. Mathematics 11. Gage, 1964

B. Elliott et al. Senior Mathematics 1. Holt, Rinehart, Winston, 1964

C. MacLean et al. Secondary School Mathematics, Grade Eleven. Copp Clark, 1964

Textbooks designed for Curriculum I.12D, Grade 10

D. Coleman et al. Mathematics 10. 2nd ed. Gage, 1967 (paper back supplements)

E. Dean et al. Contemporary Mathematics 4. Holt, Rinehart, Winston, 1967

F. MacLean et al. Secondary School Mathematics, Book Ten. Copp Clark, 1968

G. Sharp et al. Elements of Mathematics for High Schools, Book 2. 2nd ed. Ginn, 1968

Additional textbooks for Grade 10

H. Coleman et al. Mathematics 10. Gage, 1963

I. MacLean et al. Secondary School Mathematics, Grade Ten. Copp Clark, 1964

J. Sharp et al. Elements of Mathematics, Book 2. Ginn, 1963

Books that include sections on vectors and/or transformations

K. Adler, Irving. A New Look at Geometry. John Day Company, 1966 (Toronto, Longmans)

L. Adler, Irving. The New Mathematics. John Day Company, 1958. (Toronto, Longmans)

M. Coxeter. Introduction to Geometry. John Wiley, 1961. (Toronto, General Publishing)

N. Dolciani, M.P., et al. Modern Introductory Analysis. Houghton Mifflin, 1964. (Toronto, Nelson's)

O. Fletcher T.J. Some Lessons in Mathematics, Cambridge University Press, 1965. (Toronto, Macmillan)

P. Geometric Transformations Unit III (Loose Leaf). Minnemath Center, 720 Washington Avenue S.E., Minneapolis,Minn. 55414, U.S.A.

Q. Geometry, Kindergarten to Grade Thirteen. O.I.S.E. 102 Bloor Street West, Toronto 5, 1967

R. Jeger, Max. Transformation Geometry. John Wiley, 1966. (Toronto, General Publishing)

S. Kelly, Ladd. Geometry. Scott Foresman, 1965. (Toronto, Gage)

T. MacBeath, A.M. Elementary Vector Algebra. Oxford University Press, 1964

U. Mansfield, D.E.; Thompson D. Mathematics, A New Approach, Pupils' Book 3. Chatto and Windus, 1964. (Toronto, Clarke Irwin)

V. Mansfield, D.E.; Bruckheimer. Mathematics, A New Approach, Pupils' Book 4. Chatto and Windus, 1965. (Toronto, Clarke Irwin)

W. Mansfield, D.E.; Bruckheimer M. Mathematics, A New Approach, Pupils' Book 5. Chatto and Windus, 1966. (Toronto, Clarke Irwin)

X. Modern Mathematics For Schools, Book 5. Blackie and Son Ltd., (Toronto, Queenswood House Limited, 17 Prince Arthur Avenue)

Y. School Mathematics Project, Book T. (Teachers' guide available). Cambridge University Press, 1964. (Toronto, Macmillan)

Z. School Mathematics Project, Book T4. (Teachers' guide available) Cambridge University Press, 1965. (Toronto, Macmillan)

AA. S.M.P. Advanced Mathematics, Book 1. Cambridge University Press, 1967. (Toronto, Macmillan)

BB. S.M.P. Advanced Mathematics, Book 2. Cambridge University Press, 1968.

CC. The Midlands Mathematical Experiment "O" Level, Book 1. George B. Harrap, 1963. (Toronto, Clarke Irwin)

DD. The Midlands Mathematical Experiment "O" Level, Book 2. George B. Harrap, 1963. (Toronto, Clarke Irwin)

EE. The Midlands Mathematical Experiment "O" Level, Book 3. George B. Harrap, 1963. (Toronto, Clarke Irwin)

FF. Yaglom, I. Geometrical Transformations. Random House, 370 Alliance Avenue, Toronto 9, 1966

Additional References

GG. Brant, Keedy. Elementary Logic for Secondary Schools. Holt, Rinehart, Winston, 1962

HH. Crawford et al. A New Algebra for High Schools. Macmillan, 1954

II. Van Engen, et al. Algebra. Scott Foresman, 1966.(Toronto, Gage)

Articles

JJ. Adler, Irving A. "What Shall We Teach in High School Geometry"' The Mathematics Teacher, March, 1968

KK. Archambeau, Sister Mary Leona. "Pythagorean Triples Grouped into Families, The Mathematics Teacher, March, 1968

LL. Boe, Barbara L. "A Study of the Ability of Secondary School Pupils to Perceive the Plane Sections of Selected Solid Figures", The Mathematics Teacher, April, 1968

MM. Brookes, W.M. "Motion Geometry Film Loops", Mathematics Teaching, Autumn, 1967. (Loops available from Macmillan and Company, Little Essex Street, W.C.2, London, England)

NN. Dennis, Sanders. "Congruence Geometry for Junior High School", The Mathematics Teacher, April, 1968

OO. Elliott H.A. "Geometry, What Approach?", Ontario Mathematics Gazette, Vol. 4, No. 1, December, 1965

PP. Fey, James. "What's Between Q and R?", The Mathematics Teacher, April, 1967

QQ. Giles, G. "Vector Geometry in School", The Mathematical Gazette, XLVIII, February, 1964, 36-47

RR. Green, Thomas M. "The Graphing of Absolute Value Inequalities, The Mathematics Teacher, December, 1967

SS. Harris A.W. "Motion Geometry", Ontario Mathematics Gazette, Vol. 6, No. 1. December, 1967

TT. Smith, Stanley A. "What Does a Proof Really Prove", The Mathematics Teacher, May, 1968

UU. Troyer, R. "Rotations, Angles, and Trigonometry", The Mathematics Teacher, February, 1968